THE TEMPERATURE FIELD IN AN INFINITE ANISOTROPIC PRISM WITH INTERNAL HEAT RELEASE

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A solution is given to the problem of the unsteady temperature field in an infinite anisotropic prism of rectangular cross section with an internal heat source. The solution may be used to calculate the temperatures of magnetic circuits in electrical equipment.

We wish to find the unsteady temperature field in an anisotropic prism of rectangular cross section with an internal heat source q_v of uniform density. The thermal conductivities λ_x and λ_y and the heat transfer coefficients α_x and α_y are constant. The initial temperature and the temperature of the surrounding medium are zero.

Under these conditions, the equations of heat conduction are

$$\rho c \frac{\partial \vartheta(x, y, \tau)}{\partial \tau} = \lambda_x \frac{\partial^2 \vartheta(x, y, \tau)}{\partial x^2} + \lambda_y \frac{\partial^2 \vartheta(x, y, \tau)}{\partial y^2} + g_v,$$

$$0 \leqslant \tau < \infty, \quad -R_x \leqslant x \leqslant R_x, \quad -R_y \leqslant y \leqslant R_y, \quad \vartheta(x, y, 0) = 0,$$

$$\partial \vartheta(0, y, \tau)/\partial x = 0, \quad \partial \vartheta(x, 0, \tau)/\partial y = 0,$$

$$\lambda_x \frac{\partial \vartheta(R_x, y, \tau)}{\partial x} + \alpha_x \vartheta(R_x, y, \tau) = 0,$$

$$\lambda_y \frac{\partial \vartheta(x, R_y, \tau)}{\partial y} + \alpha_y \vartheta(x, R_y, \tau) = 0.$$

Making the substitution $y = \eta \sqrt{\lambda_y/\lambda_x}$ and representing the solution for $\vartheta(x, \eta, \tau)$ as $\vartheta(x, \eta, \tau) = T(x, \eta) + \theta(x, \eta, \tau)$, we obtain the two systems of equations:

$$\partial^2 T(x, \eta)/\partial x^2 + \partial^2 T(x, \eta)/\partial \eta^2 + q_v/\lambda_x = 0, \tag{1}$$

$$-R_x \leqslant x \leqslant R_x, \quad -R_y \sqrt{\lambda_x/\lambda_y} \leqslant \eta \leqslant R_y \sqrt{\lambda_x/\lambda_y},$$

$$\partial T(0, \eta)/\partial x = 0, \quad \partial T(x, 0)/\partial \eta = 0, \tag{2}$$

$$\lambda_x \frac{\partial T(R_x, \eta)}{\partial x} + \alpha_x T(R_x, \eta) = 0, \tag{3}$$

$$\lambda_y \sqrt{\lambda_x/\lambda_y} \frac{\partial T(x, R_y \sqrt{\lambda_x/\lambda_y})}{\partial \eta} + \alpha_y T(x, R_y \sqrt{\lambda_x/\lambda_y}) = 0; \tag{4}$$

$$\frac{\partial \theta(x, \eta, \tau)}{\partial \tau} = a_x \left[\frac{\partial^2 \theta(x, \eta, \tau)}{\partial x^2} + \frac{\partial^2 \theta(x, \eta, \tau)}{\partial \eta^2} \right], \tag{5}$$

$$0 \leqslant \tau < \infty, \quad -R_x \leqslant x \leqslant R_x, \quad -R_y \sqrt{\lambda_x/\lambda_y} \leqslant \eta \leqslant R_y \sqrt{\lambda_x/\lambda_y},$$

$$\theta(x, \eta, 0) = -T(x, \eta), \tag{6}$$

$$\lambda_x \frac{\partial \theta(R_x, \eta, \tau)}{\partial x} + \alpha_x \theta(R_x, \eta, \tau) = 0, \tag{7}$$

$$\lambda_y \sqrt{\frac{\lambda_x}{\lambda_y}} \frac{\partial \theta(x, R_y \sqrt{\lambda_x/\lambda_y}, \tau)}{\partial \eta} + \alpha_y \theta\left(x, R_y \sqrt{\frac{\lambda_x}{\lambda_y}}, \tau\right) = 0. \tag{8}$$

Here a_x is the thermal diffusivity in the x direction.

Solving (1)-(4), we find

$$T(x, \eta) = \frac{2 q_v R_x^2}{\lambda_x} \sum_{m=1}^{\infty} \frac{\sin \mu_m}{\mu_m^3 (1 + \sin 2\mu_m / 2\mu_m)} \left[1 - \right.$$

$$
-\frac{R_x\sqrt{\lambda_y/\lambda_x}\,\alpha_y/\lambda_y}{\mu_m}\left(\operatorname{th}\mu_m\frac{R_y}{R_x}\sqrt{\frac{\lambda_x}{\lambda_y}}+\right.
$$

$$
\left.+\frac{R_x\sqrt{\lambda_y/\lambda_x}\,\alpha_y/\lambda_y}{\mu_m}\right)^{-1}\operatorname{ch}\mu_m\frac{\eta}{R_x}\times
$$

$$
\times\left(\operatorname{ch}\mu_m\frac{R_y}{R_x}\sqrt{\frac{\lambda_y}{\lambda_x}}\right)^{-1}\cos\mu_m\frac{x}{R_x}\Bigg],
$$

where μ_m are the roots of the transcendental equation $\operatorname{ctg}\mu_m=\frac{\mu_m}{R_x\alpha_x/\lambda_x}$ given in [1].

The integral of the system (5)-(8) has the form

$$
\theta(x,\ \eta,\ \tau)=\frac{4q_vR_x^2}{\lambda_x}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu_m}{\mu_m^3(1+\sin 2\mu_m/2\mu_m)}\times
$$

$$
\times\frac{\sin\mu_n}{\mu_n(1+\sin 2\mu_n/2\mu_n)}\left[\frac{\alpha_y}{\lambda_xR_y}\ \frac{\mu_n\operatorname{ctg}\mu_n}{\mu_m^2/R_x^2+\mu_n^2\lambda_y/R_y^2\lambda_x}-1\right]\cos\mu_m\frac{x}{R_x}\times
$$

$$
\times\cos\mu_n\frac{\eta}{R_y\sqrt{\lambda_x/\lambda_y}}\ \exp\left[-a_x\left(\frac{\mu_m^2}{R_x^2}+\frac{\mu_n^2\lambda_y}{R_y^2\lambda_x}\right)\tau\right],
$$

where μ_n are the roots of the equation $\operatorname{ctg}\mu_n=\mu_n\lambda_y/R_y\alpha_y$.

Then, finally

$$
\vartheta(x,\ y,\ \tau)=\frac{2g_vR_x^2}{\lambda_x}\sum_{m=1}^{\infty}\frac{\sin\mu_m}{\mu_m^3(1+\sin 2\mu_m/2\mu_m)}\times
$$

$$
\times\Bigg\{\Bigg[1-\frac{R_x\sqrt{\lambda_y/\lambda_x}\,\alpha_y/\lambda_y}{\mu_m}\left(\operatorname{th}\mu_m\frac{R_y}{R_x}\sqrt{\frac{\lambda_x}{\lambda_y}}+\right.
$$

$$
\left.+\frac{R_x\sqrt{\lambda_y/\lambda_x}\,\alpha_y/\lambda_y}{\mu_m}\right)^{-1}\frac{\operatorname{ch}\mu_m\sqrt{\lambda_x/\lambda_y}\,y/R_x}{\operatorname{ch}\mu_m\sqrt{\lambda_x/\lambda_y}\,R_y/R_x}\Bigg]- \tag{9}
$$

$$
-2\sum_{n=1}^{\infty}\frac{\sin\mu_n}{\mu_n(1+\sin 2\mu_n/2\mu_n)}\left[1-\frac{(\mu_nR_x)^2\lambda_y}{(\mu_mR_y)^2\lambda_x+(\mu_nR_x)^2\lambda_y}\right]\times
$$

$$
\times\exp\left[-a_x\left(\frac{\mu_m^2}{R_x^2}+\frac{\mu_n^2\lambda_y}{R_y^2\lambda_x}\right)\tau\right]\cos\mu_n\frac{y}{R_y}\Bigg\}\cos\mu_m\frac{x}{R_x}.
$$

Putting $\tau=\infty$ in (9) (this corresponds to a steady temperature distribution), we obtain the solution given in [2].

REFERENCES

1. A. V. Lykov, Theory of Heat Conduction [in Russian], GTTI, 1952.
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